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Asymmetric Abelian Sandpile Models

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In the Abelian sandpile models introduced by Dhar, long-time behavior is determined by an invariant measure supported uniformly on a set of implicitly defined recurrent configurations of the system. Dhar proposed a simple procedure, the *burning algorithm*, as a possible test of whether a configuration is recurrent, and later with Majumdar verified the correctness of this test when the toppling rules of the sandpile are symmetric. We observe that the test is not valid in general and give a new algorithm which yields a test correct for all sandpiles; we also obtain necessary and sufficient conditions for the validity of the original test. The results are applied to a family of deterministic one-dimensional sandpile models originally studied by Lee, Liang, and Tzeng.

KEY WORDS: Sandpiles; Abelian sandpiles; burning algorithm; limited local models.

1. INTRODUCTION

Members of the class of cellular automata known as *sandpiles* have been much studied recently as models of "self-organized criticality" in nature. The first of these models was introduced in refs. 1 and 2, and many others were defined, classified, and studied in ref. 3. A relatively tractable class of models, the *Abelian sandpiles*, was isolated and studied by Dhar[4] (see also Section 2). Special classes of Abelian sandpiles were further investigated in refs. 5–7.

In all of these models an idealized sandpile evolves under repeated addition of grains of sand. Each added grain causes a transition of the sandpile from one stable configuration to another; during this transition, the sandpile may pass through unstable configurations, in which columns

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of sand topple and thereby transfer sand to other columns. Dhar showed that the long-time behavior of the Abelian models is determined by a simple invariant measure on the stable configurations of the system: all members of a certain set of *recurrent* configurations are equally likely. The recurrent configurations, however, are defined only implicitly. Dhar suggested that a stable configuration is in fact recurrent if and only if it passes a certain test; the test is implemented by a simple computation, called the *burning algorithm* in refs. 6 and 7. (Configurations passing this set were called *allowed* in ref. 4, but we will reserve that name for those passing a more stringent test described below.) All recurrent configurations were shown in ref. 4 to pass the burning algorithm test; conversely, it was shown in ref. 7 that if the toppling rules in the sandpile are symmetric—if the same amount of sand is transferred to site j when site i topples as is transferred to i when j topples—then all stable configurations which pass the test are recurrent. The burning algorithm test is not valid in general, however: there exist simple asymmetric sandpiles having stable configurations which pass the test but are not recurrent (see Section 4, in particular, Fig. 2).

In Section 3 we define a new algorithm, the *script algorithm*; we call the test based on this algorithm the *script test*. Sandpile configurations which pass the script test are called *allowed*, and we show that the set of recurrent configurations is precisely the set of stable configurations which are allowed in this sense. For certain sandpiles the script algorithm reduces to the burning algorithm; in Section 4 we give necessary and sufficient conditions for this to occur and show that the burning algorithm test is correct only when these conditions are satisfied. This class of sandpiles includes symmetric sandpiles, by ref. 7, and many asymmetric sandpiles as well.

Section 5 of this paper is devoted to proofs of these results. In Section 6 we discuss a class of deterministic one-dimensional sandpile models originally studied by Lee, Liang, and Tzeng.[8,9] When the dynamics of these models is expressed in terms of local slopes (differences of heights of adjacent sandpile columns) the models may in fact be regarded as asymmetric Abelian models. The test based on the burning algorithm correctly predicts recurrence for these models, and the methods of the Abelian theory may be used to show that the deterministic dynamics has a unique limit cycle.

2. DESCRIPTION OF THE MODELS

To define an *Abelian sandpile* S we begin with a finite, nonempty set V of *sites*. A *configuration* $\mathbf{z}$ of S is an assignment of a nonnegative integer z_i to each site i; z_i is normally to be thought of as the height of a column of sand at site i (although in some applications, as indicated above, it may be identified with a local slope). The configuration $\mathbf{z}$ is *stable* if each height z_i is below some threshold: $z_i \leqslant t_i$. If $\mathbf{z}$ is unstable, so that $z_k > t_k$ for some k, then the column at site k can topple, transferring sand to other columns. To picture this toppling action, it is helpful to visualize the sandpile as a directed graph $D = D(S)$, whose vertices consist of the sites of S together with an additional vertex g, the *ground.* We write Δ_{ii} for the number of edges which leave the vertex $i \in V$. A certain number, by convention denoted $-\Delta_{ij}$, of these lead to each vertex $j \in V$ which is distinct from i; the remainder of these edges (there are $\sum_{j \in V} \Delta_{ij}$ of them) lead from i to g. When the column at site k topples, one grain of sand leaves this site along each edge of D outwardly oriented from k; those grains which go to g leave the system or "fall off the table."

We take the threshold t_i to be Δ_{ii},[4] and write T_k for the toppling operator, so that $T_k\mathbf{z}$ is defined when $z_k > \Delta_{kk}$, and $(T_k\mathbf{z})_i = z_i - \Delta_{ki}$. The sandpile is thus completely specified by the set V of sites and the square matrix $\Delta = (\Delta_{ij})_{i,j \in V}$, which must satisfy (i) $\Delta_{ij} \leqslant 0$ if $i \neq j$ and (ii) $\sum_{j \in V} \Delta_{ij} \geqslant 0$.

To implement the dynamics, we will repeatedly add grains of sand to the system. If there is any site in D from which no (oriented) path leads to the ground, however, then the repeated addition of sand to that site will cause the total mass of the pile to increase without bound; we say that the sandpile is *blocked.* Here we assume that the sandpiles we discuss are not blocked unless we explicitly specify otherwise. With this understanding, any configuration $\mathbf{z}$ can evolve by repeated topplings to a stable configuration; we denote this configuration as $T\mathbf{z}$. Dhar observes[4] that the operators T_k have a weak commutativity property—$T_k T_j \mathbf{z}$ and $T_j T_k \mathbf{z}$ are both defined and are equal whenever $T_k\mathbf{z}$ and $T_j\mathbf{z}$ are defined—and that this implies that the operator T is well defined (the idea behind the proof is known to mathematicians as the "diamond lemma"[10] and goes back to a paper of Newmann[11]). If $\mathbf{z}$ is any configuration and i any site, we let $\tilde{A}_i\mathbf{z}$ be the configuration obtained from $\mathbf{z}$ by adding one grain of sand to site i $[(\tilde{A}_i\mathbf{z})_j = z_i + \delta_{ij}]$, and define $A_i\mathbf{z} = T\tilde{A}_i\mathbf{z}$ to be the resulting stable configuration. The operators A_i commute (because the T_k and $\tilde{A}_i$ jointly have the weak commutativity property), and this commutativity motivates the name "Abelian" for these sandpiles.

The dynamics of the sandpile is a Markov process on the set of stable configurations: one step of the dynamics is defined by choosing a site $i \in V$ at random and applying A_i to the current configuration—that is, adding a grain of sand at i and letting the pile stabilize. A stable configuration $\mathbf{z}$ is *recurrent* if it has nonzero weight in the (unique) invariant probability measure for this process; equivalently, $\mathbf{z}$ is recurrent if

$$\mathbf{z} = A_{i_1} \cdots A_{i_N} \mathbf{z}_{\max} \tag{1}$$

for some $i_1, \ldots, i_N$, since the maximal configuration $\mathbf{z}_{\max}$ defined by $(\mathbf{z}_{\max})_i = \Delta_{ii}$ is clearly recurrent. Dhar proved that there are precisely $\det \Delta$ recurrent configurations and that these configurations are all equally likely in the invariant measure.

We collect here some notation for configurations, which we think of as row vectors. If W is any subset of V, we write $\mathbf{e}_W$ for the configuration with $(\mathbf{e}_W)_i = 1$ if $i \in W$ and $(\mathbf{e}_W)_i = 0$ otherwise, and we write $\mathbf{e}_i$ rather than $\mathbf{e}_{\{i\}}$. In this notation, $\tilde{A}_i \mathbf{z} = \mathbf{z} + \mathbf{e}_i$ and $T_k \mathbf{z} = \mathbf{z} - \mathbf{e}_k \Delta$. We let $\mathbf{0} = \mathbf{e}_\varnothing$ and $\mathbf{1} = \mathbf{e}_V$ denote the configurations whose components are identically 0 and 1, respectively. Finally, we write $\mathbf{z}' \succ \mathbf{z}$ if $z'_i \geqslant z_i$ for all sites i, with strict inequality for at least one site; any configuration $\mathbf{z}$ satisfies $\mathbf{0} \preccurlyeq \mathbf{z}$ and a stable configuration $\mathbf{z}$ satisfies $\mathbf{z} \preccurlyeq \mathbf{z}_{\max}$.

3. THE SCRIPT ALGORITHM

We now turn to the characterization of the recurrent configurations of a sandpile S, and begin by explaining that in certain cases the first step is to decompose S into smaller sandpiles. This decomposition requires two new definitions. First, with any nonempty subset W of V we associate a new sandpile $S(W)$, whose set of sites is W and whose toppling matrix is given by the restriction of Δ to these sites. Note that the corresponding directed graph $D(S(W))$ is obtained from D by omitting the vertices not in W and all edges outgoing from them, but rerouting those edges which lead from a vertex in W to a vertex outside W so that they lead instead to g. Second, calling two sites of S equivalent if they are the same or if there is an oriented path in D from each to the other, and letting $W_1 \cdots W_m$ be the equivalence classes in V under this relation, we call $S(W_1), \ldots, S(W_m)$ the *strong components* of S (see Fig. 1); if S has only one strong component, we call S *strongly connected*. (These latter definitions correspond to the

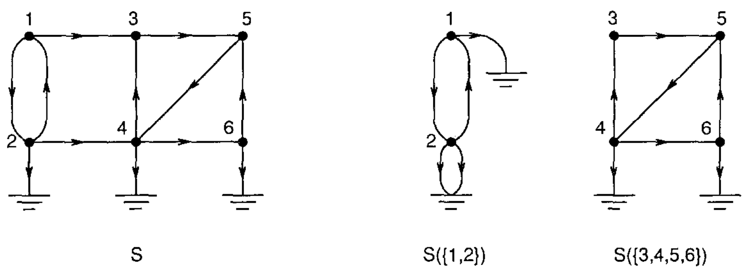

Fig. 1. A sandpile S and its strong components. Site 3 is selfish in S, but not in the strong component $S(\{3, 4, 5, 6\})$.

usual notions of strong connectivity in the directed graph obtained from D by omitting g.) It is easy to verify the following lemma, which reduces our problem to the characterization of recurrent configurations of strongly connected sandpiles.

Lemma 1. A configuration $\mathbf{z}$ of S is recurrent if and only if the restriction of $\mathbf{z}$ to each strong component $S(W_i)$ of S is a recurrent configuration of $S(W_i)$.

The key idea in testing a stable configuration $\mathbf{z}$ for recurrence is to look for appropriate configurations which topple into $\mathbf{z}$. Suppose in particular that we can find a configuration $\mathbf{z}'$ such that $\mathbf{z}' \succ \mathbf{z}$ and $T\mathbf{z}' = \mathbf{z}$. Let U be a specific (ordered) product of toppling operators T_k which implements the reduction of $\mathbf{z}'$ to $\mathbf{z}$, so that also $U\mathbf{z}' = \mathbf{z}$ or $\mathbf{z}' = U^{-1}\mathbf{z}$. Then $U^{-m}\mathbf{z}$ is defined for any positive m and $TU^{-m}\mathbf{z} = U^m U^{-m}\mathbf{z} = \mathbf{z}$; moreover, $U^{-m}\mathbf{z} \succ U^{-(m-1)}\mathbf{z} \succ \cdots \succ U^{-1}\mathbf{z} \succ \mathbf{z}$, so that $U^{-m}\mathbf{z}$ is obtained by adding at least m grains of sand to $\mathbf{z}$. Now by toppling selected columns in $U^{-m}\mathbf{z}$ we may redistribute this extra sand to other columns; specifically (see Section 5), there is a sequence of toppling operators $T_{k_1},..., T_{k_M}$ such that if m is sufficiently large,

$$\mathbf{z}'' \equiv T_{k_M} \cdots T_{k_1} U^{-m}\mathbf{z} \succcurlyeq \mathbf{z}_{\max} \tag{2}$$

Since $\mathbf{z} = TU^{-m}\mathbf{z} = T\mathbf{z}''$, $\mathbf{z}$ satisfies (1) and is recurrent.

To find a configuration like $\mathbf{z}'$ above, we focus on the *set* of topplings which make up the transformation U, ignoring for the moment the order in which they are carried out. Such a set of topplings is specified by a row vector of nonnegative integers $\mathbf{n} = (n_i)_{i \in V}$, which we will call a *script*; the script $\mathbf{n}$ is associated with any transformation $\mathbf{z} \to \mathbf{z} - \mathbf{n}\Delta$ in which, for each k, the toppling operator T_k acts n_k times. It turns out that with each strongly connected sandpile S there is associated a special script $\mathbf{N} \equiv \mathbf{N}(S)$; $\mathbf{N}$ is in a precise sense the minimal nontrivial script for which $\mathbf{N}\Delta \succ \mathbf{0}$, so that $\mathbf{z} + \mathbf{N}\Delta \succ \mathbf{z}$ for any $\mathbf{z}$. [The precise characterization of $\mathbf{N}(S)$, and a simple algorithm for its construction, are given in Lemma 7 of Section 5.] We will show that a stable configuration $\mathbf{z}$ is recurrent if and only if $(\mathbf{z} + \mathbf{N}\Delta)$ can play the role of $\mathbf{z}'$ in the previous paragraph, that is, if and only if

$$T(\mathbf{z} + \mathbf{N}\Delta) = \mathbf{z} \tag{3}$$

The criterion (3) may be restated in language closely corresponding to that of ref. 4 (see also Section 4); we make a formal definition and will verify its equivalence to (3) shortly.

Definition 2. A configuration $\mathbf{z}$ of S is *forbidden for the script* $\mathbf{n}$ if for all $i \in V$ with $n_i > 0$, $z_i \leqslant \Delta_{ii} - (\mathbf{n}\Delta)_i$. A configuration $\mathbf{z}$ is *allowed* if it is not forbidden for any script $\mathbf{n}$ satisfying $\mathbf{n} \preccurlyeq \mathbf{N}(S)$.

As in ref. 4, there is a simple test, called the *script test* and implemented by the *script algorithm*, to determine whether or not a configuration $\mathbf{z}$ is allowed. The algorithm uses a recursively defined script $\mathbf{n}$. Take initially $\mathbf{n} = \mathbf{N}$ and test whether $\mathbf{z}$ is forbidden for $\mathbf{n}$; that is, look for a site i such that

$$n_i > 0 \qquad \text{and} \qquad z_i > \Delta_{ii} - (\mathbf{n}\Delta)_i \tag{4}$$

If no such i is found, then $\mathbf{z}$ is forbidden for $\mathbf{n}$ and is not allowed. On the other hand, if i satisfies (4), then i will also satisfy (4) if $\mathbf{n}$ is replaced by any $\mathbf{n}' \prec \mathbf{n}$ for which $n'_i = n_i$; thus $\mathbf{z}$ can be forbidden only for scripts which have $n'_i < n_i$. Now decrease n_i by one—that is, replace $\mathbf{n}$ by $\mathbf{n} - \mathbf{e}_i$—and repeat. Continue until a script is encountered for which $\mathbf{z}$ is forbidden or until $\mathbf{n} = \mathbf{0}$; in the latter case, $\mathbf{z}$ is allowed.

To see that each allowed configuration $\mathbf{z}$ satisfies (3), we note that the application of the script algorithm to $\mathbf{z}$ generates a sequence $i_1, \ldots, i_K$ (with $K = \sum_{i \in V} N_i$) of sites, and corresponding scripts $\mathbf{n}_1$ $(= \mathbf{N})$, $\mathbf{n}_2, \ldots, \mathbf{n}_K$, $\mathbf{n}_{K+1}$ $(= \mathbf{0})$, where

$$z_{i_k} > \Delta_{i_k i_k} - (\mathbf{n}_k \Delta)_{i_k} \tag{5}$$

and $\mathbf{n}_{k+1} = \mathbf{n}_k - \mathbf{e}_{i_k}$. We claim that

$$T(\mathbf{z} + \mathbf{N}\Delta) = T_{i_K} T_{i_{K-1}} \cdots T_{i_1} \mathbf{z}' = \mathbf{z} \tag{6}$$

To verify (6) we need only to show that, for $k = 1, \ldots, K$, column i_k is unstable in $T_{i_{k-1}} \cdots T_{i_1}(\mathbf{z} + \mathbf{N}\Delta) = \mathbf{z} + \mathbf{n}_k \Delta$; this is precisely the content of (5).

We can summarize our discussion in a formal theorem; the remaining details of the proof are given in Section 5.

Theorem 3. Let S be a strongly connected sandpile which is not blocked. A configuration $\mathbf{z}$ of S is recurrent if and only if it is stable and allowed, that is, is stable and passes the script test.

We close this section with two peripheral remarks. First, it is natural to make the convention that blocked sandpiles have no recurrent configurations, since the total mass of the pile will increase without bound as sand is added to randomly chosen sites. The convention is consistent with the counting of recurrent configurations given above: $\det \Delta = 0$ for blocked sandpiles, since $\det \Delta$ is the number of trees in D in which each site is connected by an oriented path to g.[12] With this convention it is possible to consider blocked sandpiles on a par with all others, but this special case often requires awkward special arguments, and for this reason we will continue to assume that no sandpiles we consider are blocked unless we explicitly specify otherwise.

Second, combining Theorem 3 with Lemma 1 and the previous remark yields a characterization of recurrent configurations in general, possibly blocked, sandpiles: (i) the set of recurrent configurations of any sandpile is the product of the sets of recurrent configurations of its strong components; (ii) the set of recurrent configurations in a blocked, strongly connected sandpile is empty; and (iii) the set of recurrent configurations of a strongly connected sandpile which is not blocked consists of those configurations which pass the script algorithm. It is possible, but in practice inefficient, to unify these steps by defining a script $\mathbf{N}(S)$ for an arbitrary sandpile S in such a way that recurrent configurations of S are those which pass the script algorithm defined using this $\mathbf{N}(S)$: one simply takes $\mathbf{N}(S)$ as above if S is strongly connected and not blocked, $\mathbf{N}(S)=\mathbf{0}$ if S is strongly connected and blocked, and $\mathbf{N}(S)|_{W_i}=\mathbf{N}(S(W_i))$ if S has strong components $W_1,\ldots, W_m$.

4. COMPARISON WITH THE BURNING ALGORITHM

In this section we discuss the relation of Definition 2 to the definition of an allowed configuration given in ref. 4, which we now recall. For any subset W of V and site $i\in W$ let $d_{\text{in}}(i; W)$ denote the number of edges of the directed graph D which enter i from sites of W. We say that the restriction of a configuration $\mathbf{z}$ to a nonempty subset W is *weakly forbidden* if for all $i\in W$, $z_i\leqslant d_{\text{in}}(i; W)$, and that $\mathbf{z}$ is *weakly allowed* if no such restriction is weakly forbidden (these concepts are called simply "forbidden" and "allowed" in ref. 4). A configuration is weakly allowed if it passes the *burning algorithm*: take initially $W=V$ and look for a site $i\in W$ with

$$z_i > d_{\text{in}}(i; W) \tag{7}$$

If no such i exists, then $\mathbf{z}$ is not weakly allowed; otherwise, remove i from W and repeat. $\mathbf{z}$ is allowed if eventually W is empty.

To see the relation between the two algorithms, let us call a site i of S *selfish* if it has more incoming edges than outgoing edges in D, that is, if $d_{\text{in}}(i; V)>\Delta_{ii}$. If a strongly connected sandpile S has no selfish sites, then the sum of the rows of Δ is a vector with nonnegative components; this means that $\mathbf{N}(S)$ is the particularly simple script $\mathbf{1}$ (see Lemma 7). But then the scripts $\mathbf{n}$ with $\mathbf{n}\leqslant\mathbf{N}$ are just the scripts $\mathbf{e}_W$ for $W\subset V$, the configuration

$\mathbf{z}$ is forbidden for the script $\mathbf{e}_W$ under Definition 2 precisely when its restriction to W is weakly forbidden, and a configuration is allowed precisely when it is weakly allowed. With the identification of W with $\mathbf{e}_W$, in fact, the script algorithm reduces precisely to the burning algorithm. Thus Theorem 3 implies the following result.

Corollary 4. Let S be a strongly connected sandpile which is not blocked. If S has no selfish sites, then a configuration $\mathbf{z}$ of S is recurrent if and only if it is weakly allowed, that is, passes the burning algorithm test.

A sandpile is *symmetric* if the matrix Δ is symmetric. Since row sums of Δ are always nonnegative, symmetric sandpiles have no selfish sites and thus Corollary 4 implies that, in symmetric sandpiles, a configuration is recurrent if and only if it passes the burning algorithm. This result was obtained in ref. 7 through an ingenious correspondence between recurrent configurations and trees in the graph obtained from $D(S)$ by identifying pairs of oppositely oriented edges. In the general case there are known[12] to be $\det \Delta$ trees in $D(S)$ in which each site is connected by an oriented path to g; it would be interesting to find a similar correspondence between recurrent configurations and these trees.

On the other hand, there are many asymmetric sandpiles in which some configurations which pass the burning algorithm are not recurrent. An example is given in Fig. 2. In fact, in strongly connected sandpiles the condition that no selfish sites exist is not only sufficient but also necessary for all configurations which pass the burning algorithm to be recurrent. We state this as a theorem; the proof is given in Section 5.

Theorem 5. Let S be a strongly connected sandpile which is not blocked. Then every weakly allowed, stable configuration of S is recurrent if and only if no site of S is selfish.

The condition of strong connectedness is important in Theorem 5, for in general a sandpile may have selfish sites which are not selfish in the

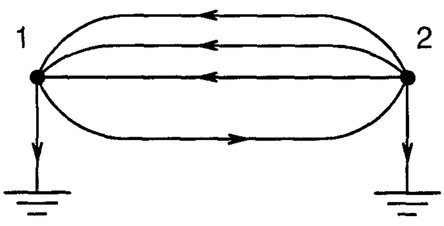

Fig. 2. A strongly connected sandpile in which site 1 is selfish. The recurrent configurations are (1, 3), (1, 4), (2, 2), (2, 3), and (2, 4); (1, 2) passes the burning algorithm test, but is not recurrent.

strong component to which they belong, and such sites do not give rise to weakly allowed stable configurations which are not recurrent (see Fig. 1). The most general statement, which follows easily from Theorem 5, is the following.

Corollary 6. Let S be any sandpile. Then every allowed, stable configuration of S is recurrent if and only if either no strong component of S contains a selfish site or some strong component of S contains no selfish site and is blocked.

5. PROOFS OF THE MAIN RESULTS

In this section we give (or complete) the proofs of the results of Sections 3 and 4. We begin by establishing the existence and some properties of the special script $\mathbf{N}(S)$.

Lemma 7. Let S be a strongly connected sandpile which is not blocked. Then there exists a unique script $\mathbf{N} \equiv \mathbf{N}(S)$ for S such that (i) $\mathbf{N} \succ \mathbf{0}$, (ii) $\mathbf{N}\Delta \succcurlyeq \mathbf{0}$, and (iii) any script $\mathbf{n}$ satisfying (i) and (ii) also satisfies $\mathbf{n} \succcurlyeq \mathbf{N}$. Moreover, (iv) $\mathbf{N}\Delta \succ \mathbf{0}$, (v) $\mathbf{N} \succcurlyeq \mathbf{1}$, and (vi) $(\mathbf{N}\Delta)_i < \Delta_{ii}$ for all i. Finally, (vii) $\mathbf{N} = \mathbf{1}$ if and only if no site of S is selfish.

Proof. We construct $\mathbf{N}$ by a "greedy" algorithm. Fix a site l and construct recursively scripts $\mathbf{0} \prec \mathbf{n}^{(1)} \prec \mathbf{n}^{(2)} \prec \cdots$, with $\mathbf{n}^{(1)} = \mathbf{e}_l$ and $\mathbf{n}^{(k)}$ for $k > 1$ defined by $\mathbf{n}^{(k)} = \mathbf{n}^{(k-1)} + \mathbf{e}_j$, where j is any site such that $(\mathbf{n}^{(k-1)}\Delta)_j < \mathbf{0}$. We claim that the algorithm eventually terminates at a script $\mathbf{n}^{(K)}$ for which $\mathbf{n}^{(K)}\Delta \succcurlyeq \mathbf{0}$. For otherwise, let W be the set of sites i for which $n_i^{(k)} \nearrow \infty$ as $k \to \infty$. Because S is not blocked, $\sum_{j \in W} \Delta_{ij} \geqslant 0$ for all $i \in W$, with strict inequality for at least one i; thus $\sum_{j \in W} (\mathbf{n}^{(k)}\Delta)_j \nearrow \infty$ as $k \to \infty$. This is a contradiction, since clearly $\mathbf{n}^{(k)}\Delta \preccurlyeq \mathbf{z}_{\max}$ for all k. We now define $\mathbf{N} \equiv \mathbf{n}^{(K)}$.

$\mathbf{N}$ clearly satisfies (i) and (ii), and any script satisfying (i) and (ii) satisfies (v), by the strong connectedness of S. If $\mathbf{n}$ also satisfies (i) and (ii), and hence (v), then we see inductively that $\mathbf{n} \succcurlyeq \mathbf{n}^{(k)}$ for all k, verifying (iii); (iii) in turn implies the uniqueness of $\mathbf{N}$. Now, $\mathbf{N}$ satisfies (iv) since $\mathbf{N}\Delta = \mathbf{0}$ is impossible because S is unblocked and hence Δ is invertible. Again inductively, each $\mathbf{n}^{(k)}$ satisfies (vi) for all i except possibly for $i = l$, and does for $i = l$ as soon as $(\mathbf{n}^{(k)})_j > 0$ for some $j \neq l$ with $\Delta_{jl} \neq 0$. Finally, if no site of S is selfish, then $\mathbf{1}$ satisfies (i)–(ii) and hence $\mathbf{N} = \mathbf{1}$ by uniqueness, while if some site is selfish, then $\mathbf{1}$ does not satisfy (ii). ∎

We can now complete the proof, begun in Section 3, of our main result.

Proof of Theorem 3. Suppose first that the configuration $\mathbf{z}$ is allowed and stable. We proved in Section 3 that this implies that $T(\mathbf{z}+\mathbf{N}\Delta)=\mathbf{z}$, and from this it follows immediately that $T(\mathbf{z}+m\mathbf{N}\Delta)=\mathbf{z}$ for any $m>0$. Since $(\mathbf{N}\Delta)_j>0$ for some $j\in V$, the column of sand at site j in the configuration $\mathbf{z}+m\mathbf{N}\Delta$ contains at least m grains of sand. It remains only to verify (2), that is, that there exists a sequence of toppling operators $T_{k_1},\ldots, T_{k_M}$ such that if m is sufficiently large,

$$\mathbf{z}''\equiv T_{k_M}\cdots T_{k_1}(\mathbf{z}+m\mathbf{N}\Delta)\succcurlyeq \mathbf{z}_{\max} \tag{8}$$

since then $\mathbf{z}$ is recurrent by (1). Now for $i\in V$ let $\rho(i)$ be the length of the shortest (oriented) path in $D(S)$ from j to i. We claim that for any q, $L\geqslant 0$ and for m sufficiently large there is a configuration

$$\mathbf{z}'''\equiv T_{k_{M'}}\cdots T_{k_1}(\mathbf{z}+m\mathbf{N}\Delta)$$

in which all sites i with $\rho(i)\leqslant q$ satisfy $(\mathbf{z}''')_i\geqslant L$. The claim is easily verified by induction on q, while (8) follows immediately from the claim by taking q maximal and $L\geqslant \max_i \Delta_{ii}$.

To establish the converse, we simply modify the proof of the similar result in ref. 4. Thus, observe from Definition 2 that if $\mathbf{z}$ is an allowed configuration, then so is $\tilde{A}_i\mathbf{z}$, and so is $T_k\mathbf{z}$, when defined. To see the latter, suppose that $\mathbf{n}\preccurlyeq\mathbf{N}$; since $\mathbf{z}$ is allowed, there exists a site i with $z_i>\Delta_{ii}-(\mathbf{n}\Delta)_i$. If $i\neq k$, then also $(T_k\mathbf{z})_i>\Delta_{ii}-(\mathbf{n}\Delta)_i$, while if $i=k$, letting $\mathbf{n}'=\mathbf{n}-\mathbf{e}_k$ and choosing i' to satisfy $z_{i'}>\Delta_{i'i'}-(\mathbf{n}'\Delta)_{i'}$, we have

$$(T_k\mathbf{z})_{i'}=z_{i'}-\Delta_{ki'}>\Delta_{ii'}-\Delta_{ki'}-(\mathbf{n}'\Delta)_i=\Delta_{ii'}-(\mathbf{n}\Delta)_i$$

In either case we see that $T_k\mathbf{z}$ is not forbidden for $\mathbf{n}$. Thus, $A_i\mathbf{z}$ is allowed for any i and any allowed $\mathbf{z}$. Since $\mathbf{z}_{\max}$ is clearly allowed, it follows from the characterization (1) that so is every recurrent configuration. ∎

We finally prove that lack of a selfish site is a necessary and sufficient condition on a strongly connected sandpile in order that the original burning algorithm correctly characterize recurrent configurations.

Proof of Theorem 5. By Corollary 4 we need only show that if S has a selfish site, then there is a configuration of S which is stable and weakly allowed but not recurrent. Let $\mathbf{z}=\mathbf{z}_{\max}-\mathbf{N}\Delta$. Now, $\mathbf{z}$ is a stable configuration, but is not allowed, since it does not satisfy (3), and hence is not recurrent. But $\mathbf{z}$ does satisfy the burning algorithm. For if not, then there is a nonempty subset W of V such that no $i\in V$ satisfies (7), or equivalently, since $d_{\text{in}}(i;\,W)=\Delta_{ii}-(\mathbf{e}_W\Delta)_i$,

$$(\mathbf{N}\Delta)_i\geqslant(\mathbf{e}_W\Delta)_i \tag{9}$$

for all $i \in W$. But (9) holds also for $i \notin W$, since the right-hand side is then nonpositive; thus $\mathbf{N}\Delta \succcurlyeq \mathbf{e}_W \Delta$. Let $\mathbf{n} = \mathbf{N} - \mathbf{e}_W$; we have shown that $\mathbf{n}\Delta \succcurlyeq \mathbf{0}$, and $\mathbf{n} \succ \mathbf{0}$ by (v) and (vii) of Lemma 7, so that by (iii) of that lemma, $\mathbf{n} \succcurlyeq \mathbf{N}$, contradicting the assumption that W is nonempty. ∎

6. A FAMILY OF DETERMINISTIC, ONE-DIMENSIONAL SANDPILES

Lee, Liang, and Tzeng[8, 9] discuss a family of one-dimensional, deterministic sandpile models (limited, nonlocal models in the terminology of ref. 3) parametrized by a positive integer m, the maximum local slope that a stable sandpile configuration will support. Intuitively, a sandpile in these models is an array of L columns of sand, with heights $h_1, h_2, \dots, h_L$; there is an infinitely high wall to the left of the first column and a table edge to the right of the Lth column. The system is driven by adding grains of sand, one at a time, with each addition increasing some column height by one. When a column becomes unstable, that is, when $h_k - h_{k+1} > m$ for some k, m grains of sand topple from it, one to each of the m columns immediately to the right—or off the edge of the table, if some of these columns do not exist. If other columns become unstable as a result, they then topple in the same fashion; the process continues until a stable configuration is reached, after which another grain is dropped. Here, as in refs. 8 and 9, we drop sand only on the first column. Under this driving mechanism the column heights may consistently be assumed to satisfy $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_L \geqslant 0$.

As described, this system is not in the class of Abelian sandpile models. Consider, however, the representation in terms of local slopes:

$$x_k = \begin{cases} h_k - h_{k+1} & \text{if } k < L \\ h_L & \text{if } k = L \end{cases}$$

Here column j topples when $x_j > m$; the resulting configuration $T_j\mathbf{x}$ is given by

$$(T_j\mathbf{x})_i = \begin{cases} x_i + m & \text{if } i = j - 1 \\ x_j - m - (1 - \delta_{iL}) & \text{if } i = j \\ x_i + 1 & \text{if } i = \min(L, j + m) \\ x_i & \text{otherwise} \end{cases}$$

This is an Abelian toppling rule, but for the first $L - 1$ columns the threshold does not satisfy the convention $t_k = \Delta_{kk}$, since $t_k = m$ for all k, but $\Delta_{kk} = m + 1$ for $1 \leqslant k \leqslant L - 1$. The configuration variable $\mathbf{z}$ given by

$z_i = x_i + 1 - \delta_{iL}$, however, does satisfy the toppling rule of a standard Abelian sandpile S. For $L > m$ the toppling matrix is

$$\Delta^{(L)} = \begin{bmatrix} m+1 & 0 & \cdots & 0 & -1 & 0 & \cdots & & \\ -m & m+1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \\ 0 & -m & m+1 & 0 & \cdots & 0 & -1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots \\ & \cdots & 0 & -m & m+1 & 0 & \cdots & 0 & -1 \\ & & \cdots & 0 & -m & m+1 & 0 & \cdots & -1 \\ & & & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \cdots & 0 & -m & m+1 & -1 \\ & & & & & \cdots & 0 & -m & m \end{bmatrix} \tag{10}$$

and the directed graph $D(S)$ is shown in Fig. 3; for $L \leqslant m$ the toppling matrix is the lower right $L \times L$ block of (10) and the directed graph is obtained by the corresponding modification of Fig. 3.

It is important to observe that the model is Abelian only when appropriately driven, that is, driven in such a way that the variables z_i increase when sand is dropped. The fundamental operator $\tilde{A}_i$ given by $\tilde{A}_i \mathbf{z} = \mathbf{z} + \mathbf{e}_i$ corresponds in the original physical model to dropping one grain of sand onto each of the first i columns of the sandpile. The driving mechanism studied in ref. 8 is just an iteration of $\tilde{A}_1$ and thus fits into the Abelian formalism, but driving by dropping sand on a single column other than the first does not.

From (10) or Fig. 3 it is clear that S has no selfish sites, so that by Corollary 4 the burning algorithm correctly predicts the recurrent configurations. These are easily seen to be precisely those observed in ref. 8: $\mathbf{z}$ is recurrent if (i) $z_k \geqslant 1$, for all k, and (ii) at least one element of the sequence $z_{L-z_L},..., z_{L-2}, z_{L-1}$, and of every sequence $z_j,..., z_{j+m-1}$ with $1 \leqslant j \leqslant L-m$, is equal to $m+1$. The number of such configurations may be

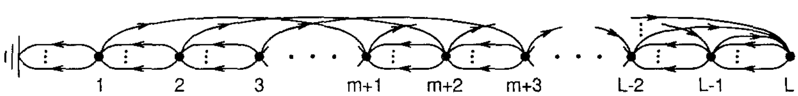

Fig. 3. The directed graph of the Abelian sandpile associated with a one-dimensional limited nonlocal model, in the case $L > m$. Here m edges join site 1 to ground and join each site i to site $i-1$, for $i > 1$; one edge joins each site i to site $i+m$ if $i+m \leqslant L$ and to site L if $L-m < i < L$, so that one edge enters site L from each of the preceding m sites.

calculated directly or obtained from (10) by expanding det Δ along the first column, leading to

$$\det \Delta^{(L)} = \begin{cases} (m+1) \det \Delta^{(L-1)} - m^{L-1} & \text{if} \quad 2 \leqslant L \leqslant m+1 \\ (m+1) \det \Delta^{(L-1)} - m^m \det \Delta^{(L-m-1)} & \text{if} \quad L > m+1 \end{cases}$$

Since $\det \Delta^{(1)} = m$, this recursion implies that $\det \Delta^{(L)} = m^L$.

Of course, the general theory implies only that these configurations are recurrent under the joint action of $A_1,..., A_L$. For the deterministic model, driven by iteration of A_1, what is relevant is the structure of orbits within the set of recurrent configurations under the action of A_1. It is shown in ref. 9 that there is in fact a single orbit, that is, that A_1 acts transitively on the recurrent configurations. We give in the next paragraph an alternative proof based on the Abelian sandpile interpretation.

Dhar[4] shows that the A_i, as operators on the recurrent configurations, form a group which is generated precisely by the commutativity relations $A_i A_j = A_j A_i$ and the relations

$$\prod_{j=1}^{L} A_j^{\Delta_{ij}} = 1, \qquad i = 1,..., L \tag{11}$$

Let p be the order of A_1 in this group; to prove transitivity of A_1, we must show that $p = m^L$. The relation $A_1^p = 1$ must be a consequence of (11), that is, there must exist a row vector $\mathbf{q}$ with integer entries such that

$$A_1^p = 1 \Leftrightarrow 1 = \prod_{i=1}^{L} \left[\prod_{j=1}^{L} A_j^{\Delta_{ij}} \right]^{q_i} = \prod_{j=1}^{L} A_j^{(\mathbf{q}\Delta)_j} \tag{12}$$

Thus, $(\mathbf{q}\Delta)_j = p\delta_{1j}$ and hence $q_i = p\Delta_{1i}^{-1}$; in particular, $q_L = p(-1)^{L+1} X/\det \Delta$, where X is the determinant of the $(L-1) \times (L-1)$ block in the upper right-hand corner of Δ. But X is relatively prime to m, since reducing all elements of this block modulo m leads to a block whose entries are all 0 or ± 1 and whose determinant involves a single product of the nonzero entries [keeping track of signs we find that $(-1)^{L+1} X \equiv 1 \bmod m$]. Since q_1 must be an integer, $p = m^L$.

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